

Appendix K: Solution of State Equations for $t_0 \neq 0$

In Section 4.11 we used the state-transition matrix to perform a transformation taking $\mathbf{x}(t)$ from an initial time, $t_0 = 0$, to any time, $t \geq 0$, as defined in Eq. (4.109). What if we wanted to take $\mathbf{x}(t)$ from a different initial time, $t_0 \neq 0$, to any time $t \geq t_0$; would Eq. (4.109) and the state-transition matrix change? To find out, we need to convert Eq. (4.109) into a form that shows $t_0 \neq 0$ as the initial state rather than $t_0 = 0$ (Kuo, 1991).

Using Eq. (4.109), we find $\mathbf{x}(t)$ at t_0 to be

$$\mathbf{x}(t_0) = \Phi(t_0)\mathbf{x}(0) + \int_0^{t_0} \Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (\text{K.1})$$

Solving for $\mathbf{x}(0)$ by premultiplying both sides of Eq. (K.1) by $\Phi^{-1}(t_0)$ and rearranging,

$$\mathbf{x}(0) = \Phi^{-1}(t_0)\mathbf{x}(t_0) - \Phi^{-1}(t_0) \int_0^{t_0} \Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (\text{K.2})$$

Substituting Eq. (K.2) into Eq. (4.109) yields

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)(\Phi^{-1}(t_0)\mathbf{x}(t_0) - \Phi^{-1}(t_0) \int_0^{t_0} \Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &\quad + \int_0^{t_0} \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &= \Phi(t)\Phi^{-1}(t_0)\mathbf{x}(t_0) - \Phi(t)\Phi^{-1}(t_0) \int_0^{t_0} \Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &\quad + \int_0^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned} \quad (\text{K.3})$$

Since $\Phi(t) = e^{At}$ and $\Phi(-t) = e^{-At}$, $\Phi(t)\Phi(-t) = \mathbf{I}$. Hence,

$$\Phi^{-1}(t) = \Phi(-t) \quad (\text{K.4})$$

Therefore

$$\Phi(t)\Phi^{-1}(t_0) = e^{At}e^{-At_0} = e^{A(t-t_0)} = \Phi(t-t_0) \quad (\text{K.5})$$

Substituting Eq. (K.5) into Eq. (K.3) yields

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t-t_0)\mathbf{x}(t_0) - \int_0^{t_0} \Phi(t-t_0)\Phi(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ &\quad + \int_0^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned} \quad (\text{K.6})$$

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But

$$\Phi(t - t_0)\Phi(t_0 - \tau) = e^{\mathbf{A}(t-t_0)}e^{\mathbf{A}(t_0-\tau)} = e^{\mathbf{A}(t-\tau)} = \Phi(t - \tau) \quad (\text{K.7})$$

Substituting Eq. (K.7) into Eq. (K.6),

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{x}(t_0) - \int_0^{t_0} \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau + \int_0^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (\text{K.8})$$

Combining the two integrals finally yields

$$\mathbf{x}(t) = \Phi(t - t_0)\mathbf{x}(t_0) + \int_{t_0}^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (\text{K.9})$$

Equation (K.9) is more general than Eq. (4.109) in that it allows us to find $\mathbf{x}(t)$ after an initial time other than $t_0 = 0$. We can see that the state-transition matrix, $\Phi(t - t_0)$, is of a more general form than previously described. In particular, the state-transition matrix is also a function of the initial time. We conclude this section by deriving some important properties of $\Phi(t - t_0)$.

Using Eq. (K.4), the inverse of $\Phi(t - t_0)$ is

$$\Phi^{-1}(t - t_0) = \Phi(t_0 - t) \quad (\text{K.10})$$

Also, from Eq. (K.7),

$$\Phi(t_2 - t_0) = \Phi(t_2 - t_1)\Phi(t_1 - t_0) \quad (\text{K.11})$$

which states that the transformation from t_0 to t_2 is the product of the transformation from t_0 to t_1 and the transformation from t_1 to t_2 .

Bibliography

Kuo, B. *Automatic Control Systems*, 6th ed. Prentice-Hall, Englewood Cliffs, NJ, 1991.

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